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HOLOMORPHIC FAMILIES OF RATIONAL MAPS

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1. INTRODUCTION

In this paper, we consider holomorphic families of rational maps from the view point of complex dynamics. The proofs of most of results presented here will be appeared elsewhere. Also, some of results which have been given already in our paper of the proceedings of the same conference last year are stated as modified version.

There is a correspondence of holomorphic families of Riemann surfaces to those of rational maps(i. e. Riemann surfaces $\sim$ quasi-Fuchsian groups $\sim$ complex dynamics). We recall the following finiteness theorem for holomorphic families of Riemann surfaces parameterized by a Riemann surface of finite type.

Theorem 1.1 (Parshin-Arakelov). *Let X be a Riemann surface of finite type. Then, there are only finitely many non-isomorphic and locally non-trivial holomorphic families of Riemann surfaces of fixed finite type (g, n) with $2g-2+n>0$ over X.*

Once the finiteness theorem is established, uniform boundedness of the number of families is an interesting question, that is, if the Riemann surface X varies in the moduli space, then whether or not there exists an upper bound of the numbers of holomorphic families over X which does not depend on X.

Recently, we have obtained a partial answer to this problem ([S2]).

Theorem 1.2. *Let X be a Riemann surface of type (p, k). Then, the number of non-isomorphic and locally non-trivial holomorphic families of Riemann surfaces of type (g, n) over X is uniformly bounded if $(g, n)=(0, n),(1,1),(1,2)$ or $(2,0)$.*

As a corresponding problem in holomorphic families of rational functions, we shall consider the following one.

Problem. Let X be a Riemann surface of type (g, n). Then, is there only finitely many non-isomorphic and locally non-trivial holomorphic families of rational maps of degree $d \geq 2$? And if it is finite, then is there an upper bound of the number of families which depends only on d, g and n?

Unfortunately, this problem has a negative answer. Namely, we can find a Riemann surface of finite type over which there are infinitely many non-isomorphic and locally non-trivial holomorphic families of rational maps. However, we shall show that if we restrict our families to ones which satisfy a certain reasonable condition, then the number of families over a Riemann surface of finite type is finite. And we establish a rigidity theorem for holomorphic families over a Riemann surface of finite type.

Secondly, we consider the monodromy of holomorphic families.

In case of holomorphic families of Riemann surfaces, all Riemann surfaces which appear in fibers of a family are quasiconformally equivalent to each other. So, analytic continuations of closed curves in the parameter space determines a (homotopy class of) quasiconformal self mapping of the Riemann surfaces. It induces a homomorphisms of the fundamental group of the surface to the mapping class group which is called monodromy. It is known that the monodromy groups play an important role in holomorphic families of Riemann surfaces (cf. [IS], [S1]).

On the other hand, in case of holomorphic families of rational maps with the same degree, all rational maps in a family are not necessarily quasiconformally equivalent to each other. In this paper, we consider holomorphic families of rational maps over the punctured disk $\Delta^* = \{0 < |z| < 1\}$, and assume that they are obtained by quasiconformal deformations. We shall show that under a condition the monodromy for a simple closed curve in Δ^* around the origin is of infinite order. It is a generalization of a result in a paper of McMullen [Mc2]. The result corresponds to the fact that for a holomorphic family of Riemann surfaces over the punctured disk, the monodromy for a simple closed curve around the origin is a Dehn twist if the family does not have an analytic continuation to the origin ([I]). Our proof deeply depends on the theory of Teichmüller spaces of complex dynamics which is developed in a paper of McMullen and Sullivan [MSu].

2. Weakly stable families and finiteness theorem

We give a finiteness theorem for holomorphic families of rational maps belonging to a certain class which is a generalization of stable (J-stable) families.

Definition 2.1 (Weakly stable family). Let $\{R_\lambda\}_{\lambda \in M}$ be a holomorphic family of rational maps of degree d over a complex manifold M and k a positive integer. Then, it is called a *weakly k-stable* family over M if it satisfies the following condition.

(1) There exists some period p such that the set of periodic points E_λ of R_λ with period p consists of just k points for every $\lambda \in M$.
(2) There exists a neighbourhood U of $\lambda \in M$ such that each points of $E_{\lambda'}$ is holomorphic with respect to $\lambda' \in U$.

We can show that any stable family is a weakly stable family.

Proposition 2.1. *Let $\{R_\lambda\}_{\lambda\in M}$ be a stable holomorphic families of rational maps of degree d over a complex manifold M. Then, it is a weakly k-stable family over M for a sufficiently large order k.*

On the other hand, there is a weakly stable family of rational maps which is not stable. Thus, the weakly stability is actually a generalization of the stability.

Theorem 2.2. *Let X be a Riemann surface of type (g, n). We denote by $N(X, d, k)$ the number of non-isomorphic and locally non-trivial weakly k-stable holomorphic families of rational maps of degree d over X, and set*

$$N(X,d) = \sum_{k=3(2d^2+1)}^{\infty} N(X,d,k).$$

Then, there exists an $N = N(g,n,d)$ depending only on g, n and d such that $N(X,d) \leq N$ for all X of type (g,n).

We show a rigidity theorem for weakly stable families of rational maps.

Theorem 2.3. *Let X be a Riemann surface of type (g, n). Then there is a number $L(g,n,d)$ which depends only on g, n and d such that if $k \geq L(g,n,d)$, then any weakly k-stable family of rational maps of degree d over X is locally trivial.*

3. Monodromy of holomorphic families of rational maps

First, we describe a motivation of the problem treated in this section. In [Mc2], McMullen considers a holomorphic family of polynomials P_λ over $\Sigma = \{\lambda \mid 10 < |\lambda| < \infty\}$ $(\cong \Delta^* = \{z \mid 0 < |z| < 1\})$ defined by

$$P_\lambda(z) = z^3 + \lambda z^2.$$

It is shown that the family is quasiconformally stable. Hence, for any point λ in Σ there exists a neighbourhood U_λ of λ in Σ such that $P_{\lambda'}$ and P_λ are quasiconformally conjugate to each other for each λ' in U_λ. Thus, we may take a quasiconformal self-mapping $f_{\lambda'}$ of $\hat{\mathbb{C}}$ such that

$$P_\lambda \circ f_{\lambda'}(z) = f_{\lambda'} \circ P_{\lambda'}(z) \tag{3.1}$$

for all $z \in \hat{\mathbb{C}}$. We take a circle $C = \{re^{i\theta} \mid 0 \leq \theta \leq 2\pi\} \subset \Sigma$. Then, an analytic continuation of P_λ along C determines a quasiconformal mapping f_C satisfying

$$P_\lambda \circ f_C(z) = f_C \circ P_\lambda(z) \tag{3.2}$$

The quasiconformal mapping f_C is not uniquely determined by C, but from (3.2) the restriction $f_C|_{J(P_\lambda)}$ on the Julia set $J(P_\lambda)$ is uniquely determined. Moreover, $f_C|_{J(P_\lambda)}$ depends on the homotopy class $[C]$ of C in Σ. So, we denote it by $\omega_{[C]}^{J(P_\lambda)}$ and call it *monodromy* of the family for $[C]$ on the Julia set. Then, the following is shown ([Mc2]).

Proposition 3.1. *The order of $\omega_{[C]}^{J(P_\lambda)}$ is infinite.*

The proof is done by using an approximation of the Julia set $J(P_\lambda)$ via a nested sequence of closed curves in the Fatou set $F(P_\lambda)$ and by a careful analysis of the action of f_C on the nested sequence. We shall use the similar method to extend Proposition 3.1.

As the first step, we derive a similar result by a different method to quasiconformal stable families of rational maps over the punctured disk Δ^*.

Let $\{R_\lambda\}_{\lambda\in\Delta^*}$ be a quasiconformally stable family over the punctured disk Δ^*. We consider a monodromy $\omega_{[C]}$ of the family for a simple closed curve C around the origin. Then, we have the following theorem.

Theorem 3.2. *Let $\{R_\lambda\}_{\lambda\in\Delta^*}$ be a locally non-trivial quasiconformally stable family over the punctured disk Δ^*. Suppose that the limit $\lim_{\lambda\to 0} R_\lambda$ does not exist in the space of rational maps which are quasiconformally conjugate to R_{λ_0} for some $\lambda_0 \in \Delta^*$. Then, the order of $\omega_{[C]}$ is infinite.*

Remark 3.1. Theorem 3.2 is applicable not only when the limit $\lim_{\lambda\to 0} R_\lambda$ does not exist but also when the limit $R_0 = \lim_{\lambda\to 0} R_\lambda$ exists but not in the Teichmüller space of R_λ.

Corollary 3.3. *Let $\{R_\lambda\}_{\lambda\in\Delta^*}$ be a locally non-trivial quasiconformally stable holomorphic family of rational maps over the punctured disk Δ^*. Suppose that R_λ ($\lambda \in \Delta^*$) is not affine and it is a post critically finite rational map, that is, the set of forward orbits of critical points of R_λ is finite. Then, the limit $\lim_{\lambda\to 0} R_\lambda$ exists in the space of rational maps which are quasiconformally conjugate to R_λ.*

The statement of Theorem 3.2 is similar to that of Proposition 3.1, but it *does not* cover that of Proposition 3.1 because $\omega_{[C]}^{J(R_\lambda)}$ may be of finite order even if $\omega_{[C]}$ is of infinite order. Here, we extend Proposition 3.1 by the following way.

Theorem 3.4. *Let $\{P_\lambda\}_{\lambda\in\Delta^*}$ be a locally non-trivial quasiconformal stable holomorphic family of polynomials of degree d over the punctured disk Δ^* and $C(\lambda)$ the set of finite critical points of P_λ. Suppose that the limit $\lim_{\lambda\to 0} P_\lambda$ does not exist in the space of rational maps which are quasiconformally conjugate to P_{λ_0} for some $\lambda_0 \in \Delta^*$, and that the set of finite critical points $C(\lambda)$ of P_λ has the following properties.*

(1) *There exists a non-empty subset $A(\lambda)$ of $C(\lambda)$ such that any $c \in A(\lambda)$ is attracted to ∞, that is, $\lim_{n\to\infty} P_\lambda^n(c) = \infty$.*
(2) *$C(\lambda) - A(\lambda)$ is not empty, and any $c \in C(\lambda) - A(\lambda)$ is either a super-attracting fixed point or belongs to a parabolic component of $F(P_\lambda)$. Furthermore, if c is in a parabolic component, then it is a unique critical point in the component.*

Then, $\omega_{[C]}^{J(P_\lambda)}$ is of infinite order.

Remark 3.2. We have some comments about the above assumption.

- From the assumption in Theorem 3.4, we have $d \geq 3$. Furthermore, we see that the Julia set $J(P_\lambda)$ is not connected.
- Suppose that in a holomorphic family of rational maps satisfying the above conditions (1) and (2), and that postcritical sets of critical points in $A(\lambda)$ are disjoint to each other for any $\lambda \in \Delta^*$. Then the postcritical set gives a holomorphic motion on Δ^*. It follows from Corollary 7.5 in [MSu] that the family is quasiconformally stable. In particular, if $A(\lambda)$ consists of only one critical point, then the family satisfying the condition (1) and (2) is quasiconformally stable.

4. Examples

In this section, we exhibit some examples about Theorem 3.4. The following families are defined over $\{|\lambda| > M\}$ for some $M > 0$. As we noted in the remark of Theorem 3.4, holomorphic motions formed by the postcritical set induces the quasiconformal stability of the family. Thus, we verify that all of the following families are quasiconformally stable if they satisfy the conditions (1) and (2) in Theorem 3.4.

Example 1. $P_\lambda(z) = z^d - \lambda z^{d-1}$ $(d \geq 3)$.

This is a direct extension of an example given in McMullen [Mc2]. We have

$$P'_\lambda(z) = dz^{d-2}\left(z - \frac{d-1}{d}\lambda\right).$$

Thus, $z = 0$ is a super-attracting fixed point and $\alpha = (d-1)\lambda/d$ is another critical point. We verify that if $M > 0$ is sufficiently large, then α is attracted to ∞. Thus, the family satisfies the condition of Theorem 3.4.

Example 2. For $n \geq 2$,

(4.1)
$$P_\lambda(z) = \frac{1}{(n+2)\lambda - n}\{(n+1)nz^{n+2} - (n+2)n(\lambda+1)z^{n+1} + (n+2)(n+1)\lambda z^n\}.$$

We have

$$P'_\lambda(z) = \frac{(n+2)(n+1)n}{(n+2)\lambda - n}z^{n-1}(z-1)(z-\lambda).$$

Hence, $z = 0, 1, \lambda$ are critical points and $z = 0, 1$ are super-attracting fixed points.

From (4.1), we have

$$|P_\lambda(\lambda)| = \frac{n(n+1)(n+2)}{|(n+2)\lambda - n|}|\lambda|^n|\lambda^2 - \lambda|.$$

Hence, if $|\lambda| \gg 1$, then $|P_\lambda(\lambda)/\lambda| \gg 1$. From

$$|P_\lambda(z)| = \frac{n(n+1)(n+2)}{|(n+2)\lambda - n|}|z|^n|(n+2)z^2 - (n+1)(\lambda+1)z + n\lambda|$$
$$\geq \frac{n(n+1)(n+2)}{|(n+2)\lambda - n|}|z|^n\{(n+2)|z|^2 - (n+1)|\lambda+1||z| - n|\lambda|\},$$

it follows that if $|z/\lambda| \gg 1$ and $|\lambda| \gg 1$, then $|P_\lambda(z)/z| \gg 1$. Hence, we verify that if $M > 0$ is sufficiently large, then λ is attracted to ∞. Thus, the example satisfies the condition of Theorem 3.4.

Finally, we give an example with a parabolic fixed point.

Example 3.

$$P_\lambda(z) = \frac{1}{3}z^3 - \frac{1}{2}(\lambda + \lambda^{-1})z^2 + z. \tag{4.2}$$

We have

$$P'_\lambda(z) = (z-\lambda)(z-\lambda^{-1}), \quad P'_\lambda(0) = 1,$$

and $P_\lambda(0) = 0$. Hence, $z = 0$ is a parabolic fixed point and $z = \lambda, \lambda^{-1}$ are critical points of $P_\lambda(z)$. We see that if $M > 0$ is sufficiently large, then $P_\lambda^n(\lambda) \to \infty$ as $n \to \infty$. On the other hand, a parabolic component contains a critical point (cf. [CG] III. Theorem 2.3). Therefore, $z = \lambda^{-1}$ is attracted to $z = 0$ and it is contained the parabolic component for $z = 0$. Thus, the example also satisfies the condition of Theorem 3.4.

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